

Higher order SUSY-QM for Pöschl-Teller potentials: coherent states and operator properties

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Abstract. This work prolongs recent investigations by Bergeron et al [see 2012 *J. Phys. A: Math. Theo.* **45** 244028] on new SUSYQM coherent states for Pöschl-Teller potentials. It mainly addresses explicit computations of eigenfunctions and spectrum associated to the higher order hierarchic supersymmetric Hamiltonian. Analysis of relevant properties and normal and anti-normal forms is performed and discussed. Coherent states of the hierarchic first order differential operator $A_{m,\nu,\beta}$ of the Pöschl-Teller Hamiltonian $\mathbf{H}_{\nu,\beta}^{(m)}$ and their characteristics are studied.

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1. Introduction

The search for exactly solvable models remains in the core of today research interest in quantum mechanics. A reference list of exactly solvable one-dimensional problems (harmonic oscillator, Coulomb, Morse, Pöschl-Teller potentials, etc.) obtained by an algebraic procedure, namely by a differential operator factorization methods [13], can be found in [1] and references therein. This technique, introduced long ago by Schrödinger [13], was analyzed in depth by Infeld and Hull [9], who made an exhaustive classification of factorizable potentials. It was reproduced rather recently in supersymmetric quantum mechanics (SUSY QM) approach [3] initiated by Witten [16] and was immediately applied to the hydrogen potential [5]. This approach gave many new exactly solvable potentials which were obtained as superpartners of known exactly solvable models. Later on, it was noticed by Witten the possibility of arranging the Schrödinger's Hamiltonians into isospectral pairs called *supersymmetric partners* [16]. The resulting supersymmetric quantum mechanics revived the study of exactly solvable Hamiltonians[15].

SUSY QM is also used for the description of hidden symmetries of various atomic and nuclear physical systems [6]. Besides, it provides a theoretical laboratory for the investigation of algebraic and dynamical problems in supersymmetric field theory. The

simplified setting of SUSY helps to analyze the difficult problem of dynamical SUSY breaking at full length and to examine the validity of the Witten index criterion[16].

The main result of the present work concerns with the explicit analytical expressions of eigenfunctions and spectrum associated to the first and second order supersymmetric Hamiltonians with Pöschl-Teller potentials. The related higher order Hamiltonian coherent states (CS) are also constructed and discussed, thus well completing recent investigations in [1] with the same model.

This paper is organized as follows. In Section 2, we recall known results and give an explicit characterization of the hierarchic Hamiltonians of the Pöschl-Teller Hamiltonian $\mathbf{H}_{\nu,\beta}$. Particular cases of eigenvalues, eigenfunctions, super-potentials and super-partner potentials are computed. In Section 3, relevant operator forms (normal and anti-normal), as well as interesting operator properties and mean-values are discussed. In Section 4, we study the CS related to the first order differential operator $A_{m,\nu,\beta}$ of the m - order hierarchic Pöschl-Teller Hamiltonian $\mathbf{H}_{\nu,\beta}^{(\mathbf{m})}$ and their main mathematical properties, i.e the orthogonality, the normalizability, the continuity in the label and the resolution of the identity. We end with some concluding remarks in Section 5.

2. The Pöschl-Teller Hamiltonian and SUSY-QM formalism

In this section, we first briefly recall the Pöschl-Teller Hamiltonian model presented in [1]. Then, we solve the associated time independent Schrödinger equation with explicit calculation of the wavefunction normalization constant. Finally, from the formalism of higher order hierarchic supersymmetric factorisation method we derive and discuss main results on the hierarchy of the Pöschl-Teller Hamiltonian.

2.1. The model

The physical system is described by the Hamiltonian [1]:

$$\mathbf{H}_{\nu,\beta}\phi := \left[-\frac{\hbar^2}{2M} \frac{d^2}{dx^2} + V_{\varepsilon_0,\nu,\beta}(x) \right] \phi \quad \text{for } \phi \in \mathcal{D}_{\mathbf{H}_{\nu,\beta}} \quad (1)$$

in a suitable Hilbert space $\mathcal{H} = L^2([0, L], dx)$ endowed with the inner product defined by

$$(u, v) = \int_0^L dx \bar{u}(x)v(x), \quad u, v \in \mathcal{H}, \quad [0, L] \subset \mathbb{R}$$

where \bar{u} denotes the complex conjugate of u . M is the particle mass and $\mathcal{D}_{\mathbf{H}_{\nu,\beta}}$ is the domain of definition of $\mathbf{H}_{\nu,\beta}$.

$$V_{\varepsilon_0,\nu,\beta}(x) = \varepsilon_0 \left(\frac{\nu(\nu+1)}{\sin^2 \frac{\pi x}{L}} - 2\beta \cot \frac{\pi x}{L} \right) \quad (2)$$

is the Pöschl-Teller potential; ε_0 is some energy scale, ν and β are dimensionless parameters.

The one-dimensional second-order operator $\mathbf{H}_{\nu,\beta}$ has singularities at the end points $x = 0$ and $x = L$ permitting to choose $\varepsilon_0 \geq 0$ and $\nu \geq 0$. Further, since the symmetry $x \rightarrow L - x$ corresponds to the parameter change $\beta \rightarrow -\beta$, we can choose $\beta \geq 0$. As assumed in [1], we consider the energy scale ε_0 as the zero point energy of the energy of the infinite well, i.e. $\varepsilon_0 = \hbar^2 \pi^2 / (2ML^2)$ so that the unique free parameters of the problem remain ν and β which will be always assumed to be positive. The case $\beta = 0$ corresponds to the symmetric repulsive potentials investigated in [2], while the case $\beta \neq 0$ leads to the Coloumb potential in the limit $L \rightarrow \infty$.

Let us define the operator $\mathcal{H}_{\nu,\beta}$ with the action $-\frac{\hbar^2}{2M}\phi''(x) + \varepsilon_0\left(\frac{\nu(\nu+1)}{\sin^2 \frac{\pi x}{L}} - 2\beta \cot \frac{\pi x}{L}\right)\phi$ with the domain being the set of smooth functions with a compact support, $\mathcal{C}_0^\infty(0, L)$. The Pöschl-Teller potential is in the limit point case at both ends $x = 0$ and $x = L$, if the parameter $\nu \geq 1/2$, and in the limit circle case at both ends if $0 \leq \nu < 1/2$. Therefore, the operator $\mathcal{H}_{\nu,\beta}$ is essentially self-adjoint in the former case. The closure of $\mathcal{H}_{\nu,\beta}$ is $\overline{\mathcal{H}_{\nu,\beta}} = \mathbf{H}_{\nu,\beta}$ i.e. $\mathcal{D}_{\overline{\mathcal{H}_{\nu,\beta}}} = \mathcal{D}_{\mathbf{H}_{\nu,\beta}}$ and its domain coincides with the maximal one, i.e.

$$\mathcal{D}_{\mathbf{H}_{\nu,\beta}} = \left\{ \phi \in ac^2(0, L), \left[-\frac{\hbar^2}{2M}\phi'' + \varepsilon_0\left(\frac{\nu(\nu+1)}{\sin^2 \frac{\pi x}{L}} - 2\beta \cot \frac{\pi x}{L}\right)\phi \right] \in \mathcal{H} \right\},$$

where $ac^2(0, L)$ denotes the absolutely continuous functions with absolutely continuous derivatives. As mentioned in [1], a function of this domain satisfies Dirichlet boundary conditions and in the range of considered ν , the deficiency indices of $\mathcal{H}_{\nu,\beta}$ is $(2, 2)$ indicating that this operator is no longer essentially self-adjoint but has a two-parameter family of self-adjoint extensions indeed. As in [1], we will restrict only to the extension described by Dirichlet boundary conditions, i. e.

$$\mathcal{D}_{\mathbf{H}_{\nu,\beta}} = \left\{ \phi \in ac^2(0, L), | \phi(0) = \phi(L), \left[-\frac{\hbar^2}{2M}\phi'' + \varepsilon_0\left(\frac{\nu(\nu+1)}{\sin^2 \frac{\pi x}{L}} - 2\beta \cot \frac{\pi x}{L}\right)\phi \right] \in \mathcal{H} \right\},$$

$\mathcal{D}_{\mathbf{H}_{\nu,\beta}}$ is dense in \mathcal{H} since $H^{2,2}(0, L) \subset \mathcal{C}_0^\infty(0, L) \subset \mathcal{D}_{\mathbf{H}_{\nu,\beta}}$ and $\mathbf{H}_{\nu,\beta}$ is self-adjoint [14] where $H^{m,n}(0, L)$ is the Sobolev space of indice (m, n) [12]. Later on, we use the dense domain:

$$\mathcal{D}_{\mathbf{H}_{\nu,\beta}} = \left\{ \phi \in AC^2(0, L), \varepsilon_0\left(\frac{\nu(\nu+1)}{\sin^2 \frac{\pi x}{L}} - 2\beta \cot \frac{\pi x}{L}\right)\phi \in \mathcal{H} \right\},$$

where $AC_{loc}^2([0, L])$ is given by

$$\begin{aligned} AC_{loc}^2([0, L]) &= \left\{ \phi \in AC([\alpha, \beta]), \forall [\alpha, \beta] \subset [0, L], [\alpha, \beta] \text{ compact} \right\}, \\ AC[\alpha, \beta] &= \left\{ \phi \in C[\alpha, \beta], \phi(x) = \phi(\alpha) + \int_{\alpha}^x dt g(t), g \in L^1([\alpha, \beta]) \right\}. \end{aligned}$$

2.2. Eigenvalues and eigenfunctions

The eigen-values $E_n^{(\nu,\beta)}$ and functions $\phi_n^{(\nu,\beta)}$ solving the Sturm-Liouville differential equation (1), i.e. $\mathbf{H}_{\nu,\beta}\phi_n^{(\nu,\beta)} = E_n^{(\nu,\beta)}\phi_n^{(\nu,\beta)}$, are given by [1]

$$E_n^{(\nu,\beta)} = \varepsilon_0 \left((n + \nu + 1)^2 - \frac{\beta^2}{(n + \nu + 1)^2} \right) \quad (3)$$

$$\phi_n^{(\nu,\beta)}(x) = K_n^{(\nu,\beta)} \sin^{\nu+n+1} \frac{\pi x}{L} \exp\left(-\frac{\beta\pi x}{L(\nu+n+1)}\right) P_n^{(a_n, \bar{a}_n)}\left(i \cot \frac{\pi x}{L}\right) \quad (4)$$

where $n \in \mathbb{N}$, $a_n = -(n + \nu + 1) + i\frac{\beta}{n+\nu+1}$, $P_n^{(\lambda, \eta)}(z)$ are the Jacobi polynomials [10] and $K_n^{(\nu,\beta)}$ is a normalization constant giving by:

$$K_n^{(\nu,\beta)} = 2^{n+\nu+1} L^{-\frac{1}{2}} \mathcal{T}(n; \nu, \beta) \mathcal{O}^{-\frac{1}{2}}(n; \nu, \beta) \exp\left(\frac{\beta\pi}{2(n+\nu+1)}\right), \quad (5)$$

where

$$\begin{aligned} \mathcal{O}(n; \nu, \beta) &= \sum_{k=0}^n \frac{(-n, -2\nu - n - 1)_k}{(-\nu - n - \frac{i\beta}{\nu+n+1})_k k! \Gamma(n + \nu + 2 - k + \frac{i\beta}{\nu+n+1})} \\ &\times \sum_{s=0}^n \frac{(-n, -2\nu - n - 1,)_s \Gamma(2n + 2\nu - s - k + 3)}{(-\nu - n + \frac{i\beta}{\nu+n+1})_s s! \Gamma(n + \nu + 2 - s - \frac{i\beta}{\nu+n+1})} \end{aligned} \quad (6)$$

and

$$\mathcal{T}(n; \nu, \beta) = n! \left| \left(-n - \nu + \frac{i\beta}{n + \nu + 1} \right)_n \right|^{-1}. \quad (7)$$

For details on the $K_n^{(\nu,\beta)}$, see Appendix A.

For $n = 0$, one can retrieve

$$\phi_0^{(\nu,\beta)}(x) = \frac{2^{\nu+1}}{\sqrt{L\Gamma(2\nu+3)}} \sin^{\nu+1} \frac{\pi x}{L} \exp\left(\frac{\beta\pi}{\nu+1} \left[\frac{1}{2} - \frac{x}{L}\right]\right). \quad (8)$$

2.3. Factorisation method and hierarchy of the Pöschl-Teller Hamiltonian: main results

Let us use the factorization method [3, 4, 8, 9] to find the hierarchy of Pöschl-Teller Hamiltonian. We assume the ground state eigenfunction $\phi_0^{(\nu,\beta)}$ and eigenvalue $E_0^{(\nu,\beta)}$ are known. Then we can define the differential operators $A_{\nu,\beta}$, $A_{\nu,\beta}^\dagger$ factorizing the Pöschl-Teller Hamiltonian $\mathbf{H}_{\nu,\beta}$ (1), and the associated superpotential $W_{\nu,\beta}$ as follows:

$$\mathbf{H}_{\nu,\beta} := \frac{1}{2M} A_{\nu,\beta}^\dagger A_{\nu,\beta} + E_0^{(\nu,\beta)}, \quad (9)$$

where the differential operators $A_{\nu,\beta}$ and $A_{\nu,\beta}^\dagger$ are defined by

$$A_{\nu,\beta} := \hbar \frac{d}{dx} + W_{\nu,\beta}(x), \quad A_{\nu,\beta}^\dagger := -\hbar \frac{d}{dx} + W_{\nu,\beta}(x), \quad (10)$$

acting in the domains

$$\mathcal{D}_{A_{\nu,\beta}} = \{\phi \in ac(0, L) \mid (\hbar\phi' + W_{\nu,\beta}\phi) \in \mathcal{H}\}, \quad (11)$$

$$\mathcal{D}_{A_{\nu,\beta}^\dagger} = \{\phi \in ac(0, L) \mid \exists \tilde{\phi} \in \mathcal{H} : [\hbar\psi(x)\phi(x)]_0^L = 0, \langle A_{\nu,\beta}\psi, \phi \rangle = \langle \psi, \tilde{\phi} \rangle, \forall \psi \in \mathcal{D}_{A_{\nu,\beta}}\},$$

where $A_{\nu,\beta}^\dagger \phi = \tilde{\phi}$. The operator $A_{\nu,\beta}^\dagger$ is the adjoint of $A_{\nu,\beta}$. Besides, considering their common restriction

$$\mathcal{D}_A = \{\phi \in AC(0, L) \mid W_{\nu,\beta}\phi \in \mathcal{H}\}, \quad (12)$$

we have $\overline{A_{\nu,\beta} \upharpoonright \mathcal{D}_A} = A_{\nu,\beta}$ and $\overline{A_{\nu,\beta}^\dagger \upharpoonright \mathcal{D}_A} = A_{\nu,\beta}^\dagger$. For more details on the role of these operators, see [1]. The super-potential $W_{\nu,\beta}$ is given by

$$W_{\nu,\beta}(x) := -\hbar \frac{[\phi_0^{(\nu,\beta)}(x)]'}{\phi_0^{(\nu,\beta)}(x)} = -\frac{\pi\hbar}{L} \left((\nu+1) \cot \frac{\pi x}{L} - \frac{\beta}{\nu+1} \right), \quad (13)$$

where $\phi_0^{(\nu,\beta)}(x)$ is defined by (8). To derive the m -th order hierarchic supersymmetric potential, we proceed as follows:

- Permut the operators $A_{\nu,\beta}^\dagger$ and $A_{\nu,\beta}$ to get the superpartner Hamiltonian $\mathbf{H}_{\nu,\beta}^{(1)}$ of $\mathbf{H}_{\nu,\beta} := \mathbf{H}_{\nu,\beta}^{(0)}$:

$$\mathbf{H}_{\nu,\beta}^{(1)} := \frac{1}{2M} A_{\nu,\beta} A_{\nu,\beta}^\dagger + E_0^{(\nu,\beta)} = -\frac{\hbar^2}{2M} \frac{d^2}{dx^2} + V_{1,\nu,\beta}(x), \quad (14)$$

where the partner-potential $V_{1,\nu,\beta}$ of $V_{\nu,\beta}$ is defined by the relation

$$V_{1,\nu,\beta}(x) := \frac{1}{2M} \left(W_{\nu,\beta}^2(x) + W_{\nu,\beta}'(x) \right) + E_0^{(\nu,\beta)}. \quad (15)$$

where $W_{\nu,\beta}$ is given by (13) and $E_0^{(\nu,\beta)}$ by (3). In the equation

$$\mathbf{H}_{\nu,\beta}^{(1)} \phi_n^{(1,\nu,\beta)} = E_n^{(1,\nu,\beta)} \phi_n^{(1,\nu,\beta)}, \quad (16)$$

the eigenfunction $\phi_n^{(1,\nu,\beta)}$ and the eigenvalue $E_n^{(1,\nu,\beta)}$ of $\mathbf{H}_{\nu,\beta}^{(1)}$ are related to those of $\mathbf{H}_{\nu,\beta}$, i.e. $E_n^{(1,\nu,\beta)} := E_{n+1}^{(\nu,\beta)}$ and $\phi_n^{(1,\nu,\beta)}(x) \propto A_{\nu,\beta} \phi_{n+1}^{(\nu,\beta)}(x)$.

Since we know $E_0^{(1,\nu,\beta)}$ and $\phi_0^{(1,\nu,\beta)}(x)$, the Hamiltonian $\mathbf{H}_{\nu,\beta}^{(1)}$ can be re-factorized to give

$$\mathbf{H}_{\nu,\beta}^{(1)} := \frac{1}{2M} A_{1,\nu,\beta}^\dagger A_{1,\nu,\beta} + E_0^{(1,\nu,\beta)}, \quad (17)$$

where

$$A_{1,\nu,\beta} := \hbar \frac{d}{dx} + W_{1,\nu,\beta}(x), \quad A_{1,\nu,\beta}^\dagger := -\hbar \frac{d}{dx} + W_{1,\nu,\beta}(x), \quad (18)$$

with

$$W_{1,\nu,\beta}(x) := -\hbar \frac{[\phi_0^{(1,\nu,\beta)}(x)]'}{\phi_0^{(1,\nu,\beta)}(x)}. \quad (19)$$

- Permut now the operators $A_{1,\nu,\beta}$ and $A_{1,\nu,\beta}^\dagger$ to build the third order hierarchic Hamiltonian $\mathbf{H}_{\nu,\beta}^{(2)}$, i.e. a superpartner Hamiltonian of $\mathbf{H}_{\nu,\beta}^{(1)}$:

$$\mathbf{H}_{\nu,\beta}^{(2)} := \frac{1}{2M} A_{1,\nu,\beta} A_{1,\nu,\beta}^\dagger + E_0^{(1,\nu,\beta)} = -\frac{\hbar^2}{2M} \frac{d^2}{dx^2} + V_{2,\nu,\beta}(x), \quad (20)$$

with

$$V_{2,\nu,\beta}(x) := \frac{1}{2M} \left(W_{1,\nu,\beta}^2(x) + \hbar W_{1,\nu,\beta}'(x) \right) + E_0^{(1,\nu,\beta)}, \quad (21)$$

where $W_{1,\nu,\beta}$ is defined in (19) and $E_0^{(1,\nu,\beta)}$ in (3). Start now from the following equation

$$\mathbf{H}_{\nu,\beta}^{(2)} \phi_n^{(2,\nu,\beta)} = E_n^{(2,\nu,\beta)} \phi_n^{(2,\nu,\beta)}.$$

The eigenvalue $E_n^{(2,\nu,\beta)}$ and the eigenfunction $\phi_n^{(2,\nu,\beta)}$ of $\mathbf{H}_{\nu,\beta}^{(2)}$ are related to those of $\mathbf{H}_{\nu,\beta}^{(1)}$, i.e. $E_n^{(2,\nu,\beta)} := E_{n+1}^{(1,\nu,\beta)}$ and $\phi_n^{(2,\nu,\beta)}(x) \propto A_{1,\nu,\beta} \phi_{n+1}^{(1,\nu,\beta)}(x)$.

From known $E_0^{(2,\nu,\beta)}$ and $\phi_0^{(2,\nu,\beta)}$, we can re-factorize the Hamiltonian $\mathbf{H}_{\nu,\beta}^{(2)}$:

$$\mathbf{H}_{\nu,\beta}^{(2)} := \frac{1}{2M} A_{2,\nu,\beta}^\dagger A_{2,\nu,\beta} + E_0^{(2,\nu,\beta)} = -\frac{\hbar^2}{2M} \frac{d^2}{dx^2} + V_{2,\nu,\beta}(x), \quad (22)$$

where the operators $A_{2,\nu,\beta}$ and $A_{2,\nu,\beta}^\dagger$ are given, respectively, by

$$A_{2,\nu,\beta} := \hbar \frac{d}{dx} + W_{2,\nu,\beta}(x), \quad A_{2,\nu,\beta}^\dagger := -\hbar \frac{d}{dx} + W_{2,\nu,\beta}(x), \quad (23)$$

with the superpotential

$$W_{2,\nu,\beta}(x) = -\hbar \frac{[\phi_0^{(2,\nu,\beta)}(x)]'}{\phi_0^{(2,\nu,\beta)}(x)}, \quad (24)$$

and the partner potential $V_{2,\nu,\beta}$

$$V_{2,\nu,\beta}(x) := \frac{1}{2M} \left(W_{2,\nu,\beta}^2(x) - \hbar W_{2,\nu,\beta}'(x) \right) + E_0^{(2,\nu,\beta)}, \quad (25)$$

where $W_{2,\nu,\beta}$ is defined in (24) and $E_0^{(2,\nu,\beta)}$ in (3).

• So, we have shown that one can determine the superpartner Hamiltonian $\mathbf{H}_{\nu,\beta}^{(1)}$ of $\mathbf{H}_{\nu,\beta}$, re-factorize $\mathbf{H}_{\nu,\beta}^{(1)}$ in order to determine its superpartner $\mathbf{H}_{\nu,\beta}^{(2)}$, then re-factorize $\mathbf{H}_{\nu,\beta}^{(2)}$ to determine its superpartner $\mathbf{H}_{\nu,\beta}^{(3)}$, and so on. Each Hamiltonian has eigenfunctions and eigenvalues. Thus, if the first Hamiltonian $\mathbf{H}_{\nu,\beta}$ has r eigenfunctions $\phi_n^{(\nu,\beta)}$ related to the eigenvalues $E_n^{(\nu,\beta)}$, $0 \leq n \leq (r-1)$, then one can always generate an hierarchy of $(r-1)$ Hamiltonians $\mathbf{H}_{\nu,\beta}^{(2)}, \mathbf{H}_{\nu,\beta}^{(3)}, \dots, \mathbf{H}_{\nu,\beta}^{(r)}$ such that $\mathbf{H}_{\nu,\beta}^{(m)}$ has the same eigenvalues as $\mathbf{H}_{\nu,\beta}$, except for the first $(m-1)$ eigenvalues of $\mathbf{H}_{\nu,\beta}$. In fact, for $m = 2, 3, 4, \dots, r$, we define the Hamiltonian in its factorized form as follows:

$$\mathbf{H}_{\nu,\beta}^{(m)} := \frac{1}{2M} A_{m,\nu,\beta}^\dagger A_{m,\nu,\beta} + E_0^{(m,\nu,\beta)} = -\frac{\hbar^2}{2M} \frac{d^2}{dx^2} + V_{m,\nu,\beta}(x), \quad (26)$$

while its super-partner Hamiltonian $\mathbf{H}_{\nu,\beta}^{(m+1)}$ is given by

$$\mathbf{H}_{\nu,\beta}^{(m+1)} := \frac{1}{2M} A_{m+1,\nu,\beta}^\dagger A_{m+1,\nu,\beta} + E_0^{(m+1,\nu,\beta)} = -\frac{\hbar^2}{2M} \frac{d^2}{dx^2} + V_{m+1,\nu,\beta}(x), \quad (27)$$

where the operators $A_{m,\nu,\beta}$ and $A_{m,\nu,\beta}^\dagger$ are defined by

$$A_{m,\nu,\beta} := \hbar \frac{d}{dx} + W_{m,\nu,\beta}(x), \quad A_{m,\nu,\beta}^\dagger := -\hbar \frac{d}{dx} + W_{m,\nu,\beta}(x), \quad (28)$$

which do not commute with the Hamiltonians $\mathbf{H}_{\nu,\beta}^{(m)}$ and $\mathbf{H}_{\nu,\beta}^{(m+1)}$, but satisfy the intertwining relations

$$\mathbf{H}_{\nu,\beta}^{(m)} A_{m,\nu,\beta}^\dagger = A_{m,\nu,\beta}^\dagger \mathbf{H}_{\nu,\beta}^{(m+1)}, \quad \mathbf{H}_{\nu,\beta}^{(m+1)} A_{m,\nu,\beta} = A_{m,\nu,\beta} \mathbf{H}_{\nu,\beta}^{(m)}. \quad (29)$$

The super-potential $W_{m,\nu,\beta}$ is given by definition by the relation:

$$W_{m,\nu,\beta}(x) := -\hbar \frac{[\phi_0^{(m,\nu,\beta)}(x)]'}{\phi_0^{(m,\nu,\beta)}(x)}, \quad (30)$$

while the potential $V_{m,\nu,\beta}$ and its superpartner potential $V_{m+1,\nu,\beta}$ are defined by

$$V_{m,\nu,\beta}(x) := \frac{1}{2M} \left(W_{m,\nu,\beta}^2(x) - \hbar W_{m,\nu,\beta}'(x) \right) + E_0^{(m,\nu,\beta)},$$

$$V_{m+1,\nu,\beta}(x) := \frac{1}{2M} \left(W_{m,\nu,\beta}^2(x) + \hbar W'_{m,\nu,\beta}(x) \right) + E_0^{(m,\nu,\beta)}. \quad (31)$$

The energy spectrum $E_n^{(m+1,\nu,\beta)}$ and the eigenfunction $\phi_n^{(m+1,\nu,\beta)}$ of the super-partner Hamiltonian $\mathbf{H}_{\nu,\beta}^{(m+1)}$ are related to those of $\mathbf{H}_{\nu,\beta}^{(m)}$, i. e. $E_n^{(m+1,\nu,\beta)} := E_{n+1}^{(m,\nu,\beta)}$ and $\phi_n^{(m+1,\nu,\beta)}(x) \propto A_{m,\nu,\beta} \phi_{n+1}^{(m,\nu,\beta)}(x)$ as formulated below.

Proposition 2.1 *The eigen-energy spectrum $E_n^{(m+1,\nu,\beta)}$ and eigen-function $\phi_n^{(m+1,\nu,\beta)}$ that solve the time-independent Schrödinger equation for the $(m+1)$ -order hierarchic superpartner Hamiltonian $\mathbf{H}_{\nu,\beta}^{(m+1)}$, i.e. $\mathbf{H}_{\nu,\beta}^{(m+1)} \phi_n^{(m+1,\nu,\beta)} = E_n^{(m+1,\nu,\beta)} \phi_n^{(m+1,\nu,\beta)}$, are given, respectively, by:*

$$E_n^{(m+1,\nu,\beta)} = \varepsilon_0 \left((n+m+\nu+2)^2 - \frac{\beta^2}{(n+m+\nu+2)^2} \right), \quad (32)$$

$$\phi_n^{(m+1,\nu,\beta)}(x) = \frac{A_{m,\nu,\beta} A_{m-1,\nu,\beta} \cdots A_{1,\nu,\beta} A_{\nu,\beta} \phi_{n+m+1}^{(\nu,\beta)}(x)}{\sqrt{(2M)^{m+1} \prod_{k=0}^m \left(E_{n+m+1}^{(\nu,\beta)} - E_k^{(\nu,\beta)} \right)}}. \quad (33)$$

As a matter of explicit computation, for the particular value of $m=0$, we get

(i) the energy spectrum

$$E_n^{(1,\nu,\beta)} = \varepsilon_0 \left((n+\nu+2)^2 - \frac{\beta^2}{(n+\nu+2)^2} \right), \quad (34)$$

(ii) the eigenfunction

$$\begin{aligned} \phi_n^{(1,\nu,\beta)}(x) = & \frac{2^{n+\nu+2} e^{\frac{\beta\pi}{2(n+\nu+2)}} \mathcal{T}(n+1; \nu, \beta)}{\sqrt{2ML(n+1)\mathcal{O}(n+1; \nu, \beta)\Delta_{n+1}^0 E_0^{(\nu,\beta)}}} \left[\left[\frac{2M(n+1)^2 \overline{\Delta_{n+1}^0 E_0^{(\nu,\beta)}}}{n+2\nu+3} \right]^{1/2} \right. \\ & \times \cos \left(\frac{\pi x}{L} - \alpha_{\nu,\beta}(n) \right) P_{n+1}^{(a_{n+1}, \bar{a}_{n+1})} \left(i \cot \frac{\pi x}{L} \right) + \frac{i\pi\hbar(n+2\nu+2)}{2L \sin \frac{\pi x}{L}} \\ & \left. \times P_n^{(a_{n+1}+1, \bar{a}_{n+1}+1)} \left(i \cot \frac{\pi x}{L} \right) \right] \sin^{\nu+n+1} \frac{\pi x}{L} \exp \left(- \frac{\beta\pi x}{L(\nu+n+2)} \right) \end{aligned} \quad (35)$$

(iii) the superpotential

$$W_{1,\nu,\beta}(x) = - \frac{\hbar\pi}{L} \left((\nu+2) \cot \frac{\pi x}{L} - \frac{\beta}{\nu+2} \right), \quad (36)$$

(iv) the potential

$$V_{1,\nu,\beta}(x) = \varepsilon_0 \left(\frac{(\nu+1)(\nu+2)}{\sin^2 \frac{\pi x}{L}} - 2\beta \cot \frac{\pi x}{L} \right), \quad (37)$$

where

$$\alpha_{\nu,\beta}(n) = \arctan \left(\frac{\beta}{(\nu+1)(\nu+n+2)} \right), \quad \overline{\Delta_{n+1}^0 E_0^{(\nu,\beta)}} := \frac{E_{n+1}^{(\nu,\beta)} - E_0^{(\nu,\beta)}}{n+1}. \quad (38)$$

It is worth noticing that the potentials $V_{\varepsilon_0,\nu,\beta}$ and $V_{m+1,\nu,\beta}$ are related in a simpler way, i.e

$$V_{m+1,\nu,\beta}(x) = V_{\varepsilon_0,\nu,\beta}(x) - \frac{\hbar^2(m+1)(2\nu+m+2)}{2M} \frac{d^2}{dx^2} \ln \left(\sin \frac{\pi x}{L} \right). \quad (39)$$

3. Relevant operator properties

This section is devoted to the investigation of relevant properties of the operators $\mathbf{H}_{\nu,\beta}^{(\mathbf{m}+1)}$ and $\mathbf{H}_{\nu,\beta}$.

Proposition 3.1 *For the operators $A_{m,\nu,\beta}$ and $A_{m,\nu,\beta}^\dagger$, there is a pair of $(m+1)$ - order hierarchic operators intertwining $\mathbf{H}_{\nu,\beta}$ and $\mathbf{H}_{\nu,\beta}^{(\mathbf{m}+1)}$, namely*

$$\mathbf{H}_{\nu,\beta} B_m^\dagger = B_m^\dagger \mathbf{H}_{\nu,\beta}^{(\mathbf{m}+1)}, \quad B_m \mathbf{H}_{\nu,\beta} = \mathbf{H}_{\nu,\beta}^{(\mathbf{m}+1)} B_m, \quad (40)$$

where

$$B_m := A_{m,\nu,\beta} \dots A_{1,\nu,\beta} A_{\nu,\beta}, \quad B_m^\dagger := A_{\nu,\beta}^\dagger A_{1,\nu,\beta}^\dagger \dots A_{m,\nu,\beta}^\dagger. \quad (41)$$

Proof. By multiplying on the left hand the intertwining relation (29) by the operator $A_{m-1,\nu,\beta}^\dagger$, we have

$$A_{m-1,\nu,\beta}^\dagger \mathbf{H}_{\nu,\beta}^{(\mathbf{m})} A_{m,\nu,\beta}^\dagger = A_{m-1,\nu,\beta}^\dagger A_{m,\nu,\beta}^\dagger \mathbf{H}_{\nu,\beta}^{(\mathbf{m}+1)}$$

which is equivalent to

$$\mathbf{H}_{\nu,\beta}^{(\mathbf{m}-1)} A_{m-1,\nu,\beta}^\dagger A_{m,\nu,\beta}^\dagger = A_{m-1,\nu,\beta}^\dagger A_{m,\nu,\beta}^\dagger \mathbf{H}_{\nu,\beta}^{(\mathbf{m}+1)}. \quad (42)$$

By continuing the process until the order $m-1$, we have

$$\mathbf{H}_{\nu,\beta}^{(1)} A_{1,\nu,\beta}^\dagger \dots A_{m,\nu,\beta}^\dagger = A_{1,\nu,\beta}^\dagger \dots A_{m,\nu,\beta}^\dagger \mathbf{H}_{\nu,\beta}^{(\mathbf{m}+1)}. \quad (43)$$

By multiplying on the left hand the equation (43) by the operator $A_{\nu,\beta}^\dagger$ we have

$$A_{\nu,\beta}^\dagger \mathbf{H}_{\nu,\beta}^{(1)} A_{1,\nu,\beta}^\dagger \dots A_{m,\nu,\beta}^\dagger = A_{\nu,\beta}^\dagger A_{1,\nu,\beta}^\dagger \dots A_{m,\nu,\beta}^\dagger \mathbf{H}_{\nu,\beta}^{(\mathbf{m}+1)}$$

which is equivalent to $\mathbf{H}_{\nu,\beta} B_m^\dagger = B_m^\dagger \mathbf{H}_{\nu,\beta}^{(\mathbf{m}+1)}$. Similarly we get $B_m \mathbf{H}_{\nu,\beta} = \mathbf{H}_{\nu,\beta}^{(\mathbf{m}+1)} B_m$. \square

Proposition 3.2 *For any positive integers n, m , the following result holds:*

$$B_n B_m^\dagger = \begin{cases} (2M)^{m+1} {}_{m+1}\Lambda_{n,\nu,\beta} \prod_{k=0}^m \left(\mathbf{H}_{\nu,\beta}^{(\mathbf{m}+1)} - E_k^{(\nu,\beta)} \right) & n > m \\ (2M)^{n+1} \prod_{k=0}^n \left(\mathbf{H}_{\nu,\beta}^{(\mathbf{n}+1)} - E_k^{(\nu,\beta)} \right)^{n+1} \Theta_{m,\nu,\beta} & n < m. \end{cases} \quad (44)$$

In particular, if $n = m$, we have

$$B_m B_m^\dagger = (2M)^{m+1} \prod_{k=0}^m \left(\mathbf{H}_{\nu,\beta}^{(\mathbf{m}+1)} - E_k^{(\nu,\beta)} \right), \quad (45)$$

$$B_m^\dagger B_m = (2M)^{m+1} \prod_{k=0}^m \left(\mathbf{H}_{\nu,\beta} - E_k^{(\nu,\beta)} \right), \quad (46)$$

where the operators ${}_{m+1}\Lambda_{n,\nu,\beta}$ and ${}^{n+1}\Theta_{m,\nu,\beta}$ are given by

$${}_{m+1}\Lambda_{n,\nu,\beta} := A_{n,\nu,\beta} A_{n-1,\nu,\beta} \dots A_{m+1,\nu,\beta}, \quad {}^{n+1}\Theta_{m,\nu,\beta} := A_{n+1,\nu,\beta}^\dagger A_{n+2,\nu,\beta}^\dagger \dots A_{m,\nu,\beta}^\dagger. \quad (47)$$

Proof. From (41), we have

$$\begin{aligned}
 B_n B_m^\dagger &= A_{n,\nu,\beta} A_{n-1,\nu,\beta} \dots (A_{\nu,\beta} A_{\nu,\beta}^\dagger) A_{1,\nu,\beta}^\dagger \dots A_{m,\nu,\beta}^\dagger \\
 &= 2M A_{m,\nu,\beta} \dots A_{1,\nu,\beta} (\mathbf{H}_{\nu,\beta}^{(1)} - E_0^{(\nu,\beta)}) A_{1,\nu,\beta}^\dagger \dots A_{m,\nu,\beta}^\dagger \\
 &= 2M A_{n,\nu,\beta} \dots A_{1,\nu,\beta} A_{1,\nu,\beta}^\dagger (\mathbf{H}_{\nu,\beta}^{(2)} - E_0^{(\nu,\beta)}) A_{2,\nu,\beta}^\dagger \dots A_{m,\nu,\beta}^\dagger \\
 &\vdots \\
 &= (2M)^{m+1} \Lambda_{n,\nu,\beta} \prod_{k=0}^m (\mathbf{H}_{\nu,\beta}^{(\mathbf{m}+1)} - E_k^{(\nu,\beta)})
 \end{aligned}$$

if $n > m$,

$$\begin{aligned}
 B_n B_m^\dagger &= A_{n,\nu,\beta} A_{n-1,\nu,\beta} \dots (A_{\nu,\beta} A_{\nu,\beta}^\dagger) A_{1,\nu,\beta}^\dagger \dots A_{m,\nu,\beta}^\dagger \\
 &= 2M A_{m,\nu,\beta} \dots A_{1,\nu,\beta} (\mathbf{H}_{\nu,\beta}^{(1)} - E_0^{(\nu,\beta)}) A_{1,\nu,\beta}^\dagger \dots A_{m,\nu,\beta}^\dagger \\
 &= 2M A_{n,\nu,\beta} \dots A_{1,\nu,\beta} A_{1,\nu,\beta}^\dagger (\mathbf{H}_{\nu,\beta}^{(2)} - E_0^{(\nu,\beta)}) A_{2,\nu,\beta}^\dagger \dots A_{m,\nu,\beta}^\dagger \\
 &\vdots \\
 &= (2M)^{n+1} \prod_{k=0}^n (\mathbf{H}_{\nu,\beta}^{(\mathbf{n}+1)} - E_k^{(\nu,\beta)})^{n+1} \Theta_{m,\nu,\beta}
 \end{aligned}$$

if $n < m$.

For $n = m$, the proof is immediate. \square

Corollary 3.3 *The operators ${}_{m+1}\Lambda_{n,\nu,\beta}$ and ${}^{n+1}\Theta_{m,\nu,\beta}$ satisfies the following identities*

$${}_{m+1}\Lambda_{n,\nu,\beta} {}_{m+1}\Lambda_{n,\nu,\beta}^\dagger = (2m)^{n-m} \prod_{k=m+1}^n (\mathbf{H}_{\nu,\beta}^{(\mathbf{n}+1)} - E_k^{(\nu,\beta)}), \quad (48)$$

$${}_{m+1}\Lambda_{n,\nu,\beta}^\dagger {}_{m+1}\Lambda_{n,\nu,\beta} = (2m)^{n-m} \prod_{k=m+1}^n (\mathbf{H}_{\nu,\beta}^{(\mathbf{m}+1)} - E_k^{(\nu,\beta)}), \quad (49)$$

$${}^{n+1}\Theta_{m,\nu,\beta} {}^{n+1}\Theta_{m,\nu,\beta}^\dagger = (2m)^{m-n} \prod_{k=n+1}^m (\mathbf{H}_{\nu,\beta}^{(\mathbf{n}+1)} - E_k^{(\nu,\beta)}), \quad (50)$$

$${}^{n+1}\Theta_{m,\nu,\beta}^\dagger {}^{n+1}\Theta_{m,\nu,\beta} = (2m)^{m-n} \prod_{k=n+1}^m (\mathbf{H}_{\nu,\beta}^{(\mathbf{m}+1)} - E_k^{(\nu,\beta)}). \quad (51)$$

Proof. The proof is obviously true by using (29) and (47). \square

Besides, considering the supercharges

$$Q := \begin{pmatrix} 0 & 0 \\ B_m & 0 \end{pmatrix}, \quad Q^\dagger := \begin{pmatrix} 0 & B_m^\dagger \\ 0 & 0 \end{pmatrix}.$$

and the SUSY Hamiltonian $\mathbf{H}_{\nu,\beta}^{\text{ss}}$ given by

$$\mathbf{H}_{\nu,\beta}^{\text{ss}} := (2M)^{m+1} \begin{bmatrix} \prod_{k=0}^m (\mathbf{H}_{\nu,\beta} - E_k^{(\nu,\beta)}) & 0 \\ 0 & \prod_{k=0}^m \left(\mathbf{H}_{\nu,\beta} - E_k^{(\nu,\beta)} + \frac{\varepsilon_0(m+1)(2\nu+m+2)}{\sin^2 \frac{\pi x}{L}} \right) \end{bmatrix}, \quad (52)$$

we readily check, like in [4, 8, 16], that

$$\begin{aligned}\mathbf{H}_{\nu,\beta}^{\text{ss}} &= \{Q, Q^\dagger\} := QQ^\dagger + Q^\dagger Q, \quad [\mathbf{H}_{\nu,\beta}^{\text{ss}}, Q] := \mathbf{H}_{\nu,\beta}^{\text{ss}} Q - Q \mathbf{H}_{\nu,\beta}^{\text{ss}} = 0 \\ [\mathbf{H}_{\nu,\beta}^{\text{ss}}, Q^\dagger] &:= \mathbf{H}_{\nu,\beta}^{\text{ss}} Q^\dagger - Q^\dagger \mathbf{H}_{\nu,\beta}^{\text{ss}} = 0.\end{aligned}\quad (53)$$

In terms of the Hermitian supercharges

$$Q_1 := \frac{1}{2} \begin{pmatrix} 0 & B_m^\dagger \\ B_m & 0 \end{pmatrix} \quad \text{and} \quad Q_2 := \frac{1}{2i} \begin{pmatrix} 0 & B_m^\dagger \\ -B_m & 0 \end{pmatrix}$$

the superalgebra (53) takes the form

$$[Q_i, \mathbf{H}_{\nu,\beta}^{\text{ss}}] = 0, \quad \{Q_i, Q_j\} := Q_i Q_j + Q_j Q_i = \delta_{ij} \mathbf{H}_{\nu,\beta}^{\text{ss}}, \quad i, j = 1, 2. \quad (54)$$

Proposition 3.4 *The actions of B_m^\dagger and B_m on the normalized eigenfunctions $\phi_n^{(m+1,\nu,\beta)}$ and $\phi_n^{(\nu,\beta)}$ of $\mathbf{H}_{\nu,\beta}^{(m+1)}$ and $\mathbf{H}_{\nu,\beta}$, associated to the eigenvalues $E_n^{(m+1,\nu,\beta)}$ and $E_n^{(\nu,\beta)}$, are given by*

$$\begin{aligned}B_m^\dagger \phi_n^{(m+1,\nu,\beta)}(x) &= 2^{n+m+\nu+2} (\hbar\pi L^{-1})^{m+1} \mathcal{T}(n+m+1; \nu, \beta) \left[L \mathcal{O}(n+m+1; \nu, \beta) \right]^{-\frac{1}{2}} \\ &\quad \times \exp \left[\frac{\beta\pi}{n+m+\nu+2} \left(\frac{1}{2} - \frac{x}{L} \right) \right] \mathcal{M}(n, m; \nu, \beta) \sin^{n+m+\nu+2} \left(\frac{\pi x}{L} \right) \\ &\quad \times P_{n+m+1}^{(a_{n+m+1}, \bar{a}_{n+m+1})} \left(i \cot \frac{\pi x}{L} \right),\end{aligned}\quad (55)$$

and

$$B_m \phi_{n+m+1}^{(\nu,\beta)}(x) = (\pi \hbar L^{-1})^{m+1} \mathcal{M}(n, m; \nu, \beta) \phi_n^{(m+1,\nu,\beta)}(x), \quad (56)$$

respectively, where $\mathcal{M}(n, m; \nu, \beta)$ is expressed by

$$\mathcal{M}^2(n, m; \nu, \beta) = \prod_{k=0}^m \frac{(n+m-k+1)}{(n+m+2\nu+k+3)^{-1}} \left(1 + \frac{\beta^2}{[(k+\nu+1)(n+m+\nu+2)]^2} \right). \quad (57)$$

Proof. The proof is immediate by using (33) and (46). \square

Proposition 3.5 *Consider $|\phi_n^{(\nu,\beta)}\rangle$ and $|\phi_n^{(m+1,\nu,\beta)}\rangle$ two states in the Hilbert space \mathcal{H} . The operators $B_m B_m^\dagger$ and $B_m^\dagger B_m$ mean-values are given by*

$$\langle B_m B_m^\dagger \rangle_{\phi_n^{(m+1,\nu,\beta)}} = \left[(\pi \hbar L^{-1})^{m+1} \mathcal{M}(n, m; \nu, \beta) \right]^2, \quad (58)$$

$$\langle B_m^\dagger B_m \rangle_{\phi_n^{(\nu,\beta)}} = \left[(\pi \hbar L^{-1})^{m+1} \mathcal{M}(n-m-1, m; \nu, \beta) \right]^2, \quad (59)$$

where $\langle A_{\nu,\beta} \rangle_{\phi_n^{(\nu,\beta)}} := \int_0^L dx \overline{\phi_n^{(\nu,\beta)}(x)} A_{\nu,\beta} \phi_n^{(\nu,\beta)}(x)$.

Proof. It uses Proposition 3.1. \square

Corollary 3.6 *The operators ${}_{m+1}\Lambda_{n,\nu,\beta}$ and ${}^{n+1}\Theta_{m,\nu,\beta}$ satisfy the following identities:*

$$\langle {}_{m+1}\Lambda_{n,\nu,\beta} {}_{m+1}\Lambda_{n,\nu,\beta}^\dagger \rangle_{\phi_n^{(n+1,\nu,\beta)}} = \left[(\hbar\pi L^{-1})^{n-m} \right]^2 \mathcal{N}(n, n; \nu, \beta) \mathcal{N}^{-1}(n, m; \nu, \beta), \quad (60)$$

$$\langle {}_{m+1}\Lambda_{n,\nu,\beta}^\dagger {}_{m+1}\Lambda_{n,\nu,\beta} \rangle_{\phi_n^{(m+1,\nu,\beta)}} = \left[(\hbar\pi L^{-1})^{n-m} \mathcal{M}(m, n; \nu, \beta) \mathcal{M}^{-1}(n, m; \nu, \beta) \right]^2, \quad (61)$$

$$\langle {}^{n+1}\Theta_{m,\nu,\beta} {}^{n+1}\Theta_{m,\nu,\beta}^\dagger \rangle_{\phi_n^{(n+1,\nu,\beta)}} = \left[(\hbar\pi L^{-1})^{m-n} \right]^2 \mathcal{N}^{-1}(n, n; \nu, \beta) \mathcal{N}(n, m; \nu, \beta), \quad (62)$$

$$\langle {}^{n+1}\Theta_{m,\nu,\beta}^\dagger {}^{n+1}\Theta_{m,\nu,\beta} \rangle_{\phi_n^{(m+1,\nu,\beta)}} = \left[(\hbar\pi L^{-1})^{m-n} \mathcal{M}(n, m; \nu, \beta) \mathcal{M}^{-1}(m, n; \nu, \beta) \right]^2, \quad (63)$$

where $\mathcal{N}(n, m; \nu, \beta)$ is given by

$$\mathcal{N}(n, m; \nu, \beta) = \prod_{k=0}^m \frac{(2n - k + 1)}{(2n + 2\nu + k + 3)^{-1}} \left(1 + \frac{\beta^2}{[(k + \nu + 1)(2n + \nu + 2)]^2} \right) \quad (64)$$

and $\mathcal{N}(n, n; \nu, \beta) = \mathcal{M}^2(n, n; \nu, \beta)$.

Remark that the equations (62) and (63) can be obtained by replacing \mathcal{N} and n, m by \mathcal{N}^{-1} and m, n in (60) and (61), respectively.

4. Coherents states

Let $|\zeta_z^{[m,\nu,\beta]}\rangle$, $z \in \mathbb{C}$ be the eigenstates of the operator $A_{m,\nu,\beta}$ associated to the eigenvalue z . Then,

$$|\zeta_z^{[m,\nu,\beta]}\rangle = \mathcal{R} \exp\left(\frac{zx}{\hbar}\right) \phi_0^{(m,\nu,\beta)}(x), \quad \forall x \in [0, L], \quad (65)$$

where \mathcal{R} is the normalization constant. In order to determine \mathcal{R} , let us consider the set $\mathcal{K} = \{(q, p) | q \in [0, L], p \in \mathbb{R}\}$ which corresponds to the classical phase space of the Pöschl-Teller problem. We re-express the operator $\mathbf{A}_{m,\nu,\beta}$ in terms of \mathbf{Q} and \mathbf{P} i.e $\mathbf{A}_{m,\nu,\beta} = W_{m,\nu,\beta}(\mathbf{Q}) + i\mathbf{P}$, where their actions on the function ϕ are given by $\mathbf{Q} : \phi(x) \rightarrow x\phi(x)$ and $\mathbf{P} : \phi(x) \rightarrow -i\hbar\phi'(x)$ on \mathcal{D}_A . Latter on, we change the variable z as $z = W_{m,\nu,\beta}(q) + ip$ [1, 2] i.e $|\zeta_{W_{m,\nu,\beta}(q)+ip}^{[m,\nu,\beta]}\rangle = |\eta_{q,p}^{[m,\nu,\beta]}\rangle$. Then, the equation (65) becomes

$$|\eta_{q,p}^{[m,\nu,\beta]}\rangle = \mathcal{R}_m^{(\nu,\beta)}(q) \exp\left(\frac{(W_{m,\nu,\beta}(q) + ip)x}{\hbar}\right) \phi_0^{(m,\nu,\beta)}(x), \quad \forall x \in [0, L], \quad (66)$$

where $\phi_0^{(m,\nu,\beta)}$ is given in (33). The normalization constant $\mathcal{R}_m^{(\nu,\beta)}(q)$ is given by

$$\mathcal{R}_m^{(\nu,\beta)}(q) = \exp\left(-\frac{LW_{m,\nu,\beta}(q)}{2\hbar}\right) \tilde{\mathcal{O}}_m(q; L; \nu, \beta), \quad (67)$$

where $\tilde{\mathcal{O}}_m(q; L; \nu, \beta)$ is provided by the expression

$$\begin{aligned} \tilde{\mathcal{O}}_m^2(q; L; \nu, \beta) &= \sum_{k=0}^m \frac{(-m, -m - 2\nu - 1)_k}{(-m - \nu - \frac{i\beta}{\nu+m+1})_k k! \Gamma(m + \nu + 2 - k + \frac{i\beta}{\nu+m+1})} \\ &\times \sum_{s=0}^m \frac{(-m, -m - 2\nu - 1)_s \Gamma(2m + 2\nu - s - k + 3)}{(-m - \nu + \frac{i\beta}{\nu+m+1})_s s! \Gamma(m + \nu + 2 - s - \frac{i\beta}{\nu+m+1})} \\ &\times \left[\sum_{k=0}^m \frac{(-m, -2\nu - m - 1)_k}{(-m - \nu - \frac{i\beta}{\nu+m+1})_k \Gamma(m + \nu + 2 - k + i(\nu + m + 1) \cot \frac{\pi q}{L}) k!} \right] \end{aligned}$$

$$\times \sum_{s=0}^m \frac{(-m, -2\nu - m - 1)_s \Gamma(2m + 2\nu - s - k + 3)}{(-m - \nu + \frac{i\beta}{\nu+m+1})_s \Gamma(m + \nu + 2 - s - i(\nu + m + 1) \cot \frac{\pi q}{L}) s!} \Big]^{-1} \quad (68)$$

For computational details, see Appendix C.

In the limit, when the parameter $m \rightarrow 0$, the coherent states (66), (67) are reduced to ones obtained by Bergeron et al [1].

The scalar product of two coherent states $|\eta_{q,p}^{[m,\nu,\beta]}\rangle$ and $|\eta_{q',p'}^{[m,\nu',\beta']}\rangle$ satisfies

$$\langle \eta_{q',p'}^{[m,\nu',\beta']} | \eta_{q,p}^{[m,\nu,\beta]} \rangle = L e^{\frac{L\alpha}{2\hbar}} \mathcal{R}_m^{(\nu',\beta')}(q') K_m^{(\nu',\beta')}(q) K_m^{(\nu,\beta)}(q) \tilde{\mathcal{T}}(m; \nu, \nu', \beta, \beta', \frac{L\alpha}{2\pi\hbar}), \quad (69)$$

where $\alpha = W_{m,\nu,\beta}(q) + W_{m,\nu',\beta'}(q') + i(p - p')$ and

$$\begin{aligned} \tilde{\mathcal{T}}(m; \nu, \nu', \beta, \beta', \frac{L\alpha}{2\pi\hbar}) &= \left(-m - \nu - i \frac{\beta}{m + \nu + 1}, -m - \nu' + i \frac{\beta'}{m + \nu' + 1} \right)_m \\ &\times \sum_{k=0}^m \frac{2^{-2m-\nu-\nu'-2} (-m, -m - \nu - \nu' - 1)_k}{(-\nu - m - i \frac{\beta}{\nu+m+1})_k \Gamma(m + \frac{\nu+\nu'}{2} + 2 - k - i \frac{L\alpha}{2\pi\hbar}) k! m!} \\ &\times \sum_{s=0}^m \frac{(-m, -m - \nu - \nu' - 1)_s \Gamma(2m + \nu + \nu' + 3 - k - s)}{(-m - \nu' + i \frac{\beta'}{\nu'+m+1})_s \Gamma(m + \frac{\nu+\nu'}{2} + 2 - s + i \frac{L\alpha}{2\pi\hbar}) s! m!}. \end{aligned} \quad (70)$$

Proposition 4.1 *The coherent states defined in (66)*

(i) *are normalized*

$$\langle \eta_{q,p}^{[m,\nu,\beta]} | \eta_{q,p}^{[m,\nu,\beta]} \rangle = 1, \quad (71)$$

(ii) *are not orthogonal to each other, i.e.*

$$\langle \eta_{q',p'}^{[m,\nu',\beta']} | \eta_{q,p}^{[m,\nu,\beta]} \rangle \neq \delta(q - q') \delta(p - p'), \quad (72)$$

(iii) *are continuous in q, p ,*

(iv) *solve the identity, i.e.*

$$\int_{\mathcal{K}} \frac{dq dp}{2\pi\hbar} |\eta_{q,p}^{[m,\nu,\beta]}\rangle \langle \eta_{q,p}^{[m,\nu,\beta]}| = 1. \quad (73)$$

Proof.

• Non orthogonality: From (69) one can see that

$$\langle \eta_{q',p'}^{[m,\nu',\beta']} | \eta_{q,p}^{[m,\nu,\beta]} \rangle \neq 0, \quad (74)$$

which signifies that the CS are not orthogonal.

• In the limit when the parameters $\nu' \rightarrow \nu, \beta' \rightarrow \beta, q' \rightarrow q$ and $p' \rightarrow p$, the quantity

$$L e^{\frac{L\alpha}{2\hbar}} \tilde{\mathcal{T}}(m; \nu, \nu', \beta, \beta', \frac{L\alpha}{2\pi\hbar}) \rightarrow \frac{1}{(\mathcal{R}_m^{(\nu,\beta)}(q) K_m^{(\nu,\beta)})^2} \text{ and } \langle \eta_{q,p}^{[m,\nu,\beta]} | \eta_{q,p}^{[m,\nu,\beta]} \rangle = 1,$$

i.e the CS are normalized.

• Continuity in q, p

$$||(\eta_{q',p'}^{[m,\nu,\beta]} - \eta_{q,p}^{[m,\nu,\beta]})||^2 = 2 \left(1 - \mathcal{R}_e \langle \eta_{q',p'}^{[m,\nu,\beta]} | \eta_{q,p}^{[m,\nu,\beta]} \rangle \right). \quad (75)$$

So, $||(\eta_{q',p'}^{[m,\nu,\beta]} - \eta_{q,p}^{[m,\nu,\beta]})||^2 \rightarrow 0$ as $|q' - q|, |p' - p| \rightarrow 0$, since $\langle \eta_{q',p'}^{[m,\nu,\beta]} | \eta_{q,p}^{[m,\nu,\beta]} \rangle \rightarrow 1$ as $|q' - q|, |p' - p| \rightarrow 0$.

• Resolution of the identity

Here we proceed as in [1] to show that

$$\begin{aligned} & \int_{\mathbb{R}} \frac{4^{\nu+m} \Gamma(m+\nu+1-k+i(m+\nu+1)\cot\pi q) \Gamma(m+\nu+1-s-i(m+\nu+1)\cot\pi q)}{\pi^2 \Gamma(2m+2\nu-k-s+2)} \\ & \quad \times \exp\left((\nu+m+1)\cot\pi q(1-2x)\right) dq \\ & = \frac{1}{\sin^{2\nu+2m+2}(\pi x) \left(\frac{1+i\cot(\pi x)}{2}\right)^k \left(\frac{1-i\cot(\pi x)}{2}\right)^s}, \quad \forall x \in]0, 1[, \quad \forall \nu > -1. \end{aligned} \quad (76)$$

Let $\phi \in \mathcal{H}$ and $h_{q,p}$ a function in $L^2(\mathbb{R}, dx)$ defined by

$$h_{q,p}(x) := \begin{cases} \phi(x) \exp\left(\frac{W_{m,\nu,\beta}(q)+ip}{\hbar}x\right) \phi_0^{(m,\nu,\beta)}(x), & \text{if } x \in [0, L] \\ 0 & \text{otherwise.} \end{cases} \quad (77)$$

One can see that the scalar product $\langle \eta_{q,p}^{[m,\nu,\beta]} | \phi \rangle$ given by

$$\langle \eta_{q,p}^{[m,\nu,\beta]} | \phi \rangle = \mathcal{R}_m^{(\nu,\beta)}(q) \int_0^L dx e^{-i\frac{p}{\hbar}x} \phi(x) \exp\left(\frac{W_{m,\nu,\beta}(q)x}{\hbar}\right) \overline{\phi_0^{(m,\nu,\beta)}(x)} \quad (78)$$

is the Fourier transform of $h_{q,p}$, i.e. $\langle \eta_{q,p}^{[m,\nu,\beta]} | \phi \rangle = \mathcal{R}_m^{(\nu,\beta)}(q) \hat{h}_{q,p}(p/\hbar)$. Since the function $h_{q,p} \in L^1(\mathbb{R}, dx) \cap L^2(\mathbb{R}, dx)$, by using the Plancherel-Parseval Theorem (PPT) we have

$$\int_{\mathbb{R}} \frac{dp}{2\pi\hbar} |\langle \eta_{q,p}^{[m,\nu,\beta]} | \phi \rangle|^2 = (\mathcal{R}_m^{(\nu,\beta)}(q))^2 \int_0^L \frac{dx}{4\pi^2} |h_{q,p}(x)|^2. \quad (79)$$

The Fubini theorem yields

$$\int_{\mathcal{K}} \frac{dq dp}{2\pi\hbar} |\langle \eta_{q,p}^{[m,\nu,\beta]} | \phi \rangle|^2 = \int_0^L dx \int_0^L \frac{dq}{4\pi^2} (\mathcal{R}_m^{(\nu,\beta)}(q))^2 |\phi(x)|^2 e^{\frac{2W_{m,\nu,\beta}(q)x}{\hbar}} \overline{\phi_0^{(m,\nu,\beta)}(x)} \phi_0^{(m,\nu,\beta)}(x) \quad (80)$$

After using the inverse Fourier transform (see Appendix D), the above equation yields

$$\int_{\mathcal{K}} \frac{dq dp}{2\pi\hbar} |\langle \eta_{q,p}^{[m,\nu,\beta]} | \phi \rangle|^2 = \int_0^L dx |\phi(x)|^2. \quad (81)$$

By using the polarization identity on the interval $[0, L]$, i.e. $\int_0^L dx |\psi(x)|^2 = \int_0^L dx \langle \psi | x \rangle \langle x | \psi \rangle$ we get the resolution of the identity. \square

5. Conclusion

In this paper, we have determined a family of normalized eigenfunctions of the hierarchic Hamiltonians of the Pöschl-Teller Hamiltonian $\mathbf{H}_{\nu,\beta}$. New operators with novel relevant properties and their mean values are determined. A new hierarchic family of CS is determined and discussed. In the limit, when $m \rightarrow 0$, the constructed CS well reduce to the CS investigated by Bergeron et al [1].

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Appendix A. The normalization constant of the eigenvector $|\phi_n^{(\nu,\beta)}\rangle$

By using the property of the eigenstates, we have

$$\begin{aligned}
\delta_{n,m} &= (\phi_n^{(\nu,\beta)}, \phi_m^{(\nu,\beta)}) \\
&= \overline{K_n^{(\nu,\beta)}} K_m^{(\nu,\beta)} \int_0^L dx \sin^{2\nu+n+m+2} \frac{\pi x}{L} e^{-\frac{\beta\pi x}{L} \left(\frac{1}{\nu+n+1} + \frac{1}{\nu+m+1} \right)} \\
&\times \overline{P_n^{(a_n, \bar{a}_n)} \left(i \cot \frac{\pi x}{L} \right)} P_m^{(a_m, \bar{a}_m)} \left(i \cot \frac{\pi x}{L} \right) \\
&= \overline{K_n^{(\nu,\beta)}} K_m^{(\nu,\beta)} \frac{(\bar{a}_n + 1)_n (a_m + 1)_n}{n! m!} \sum_{k=0}^m \frac{(-m, a_m + \bar{a}_m + m + 1)_k}{(\bar{a}_m + 1)_k k!} \\
&\times \sum_{s=0}^n \frac{(-n, a_n + \bar{a}_n + n + 1)_s}{(a_n + 1)_s s!} \times \mathcal{J}, \tag{82}
\end{aligned}$$

where

$$\mathcal{J} = 2^{-k-s} \int_0^L dx \left[\sin^{2\nu+m+n+2} \frac{\pi x}{L} e^{-\frac{\beta\pi x}{L} \left(\frac{1}{\nu+n+1} + \frac{1}{\nu+m+1} \right)} \left(1 + i \cot \frac{\pi x}{L} \right)^k \left(1 - i \cot \frac{\pi x}{L} \right)^s \right]$$

In [1] it is shown that

$$\int_0^1 dx \sin^{2\delta+2}(\pi x) e^{zx} = \frac{\Gamma(2\delta+3) e^{z/2}}{4^{\delta+1} \Gamma(\delta+2+i\frac{z}{2\pi}) \Gamma(\delta+2-i\frac{z}{2\pi})}, \quad \delta > -\frac{3}{2}.$$

Therefore,

$$\begin{aligned}
\frac{\delta_{n,m}}{\overline{K_n^{(\nu,\beta)}} K_m^{(\nu,\beta)}} &= L \frac{(-\nu - m - \frac{i\beta}{\nu+m+1})_m (-\nu - n + \frac{i\beta}{\nu+n+1})_n}{n! m! \exp \left\{ -\frac{\beta\pi}{2} \left(\frac{1}{\nu+n+1} + \frac{1}{\nu+m+1} \right) \right\} 2^{2\nu+n+m+2}} \\
&\times \sum_{k=0}^m \frac{(-m, -2\nu - m - 1)_k}{(-\nu - m - \frac{i\beta}{\nu+m+1})_k k! \Gamma(\frac{n+m}{2} + \nu + 2 - k + \frac{i\beta}{\nu+n+1})} \\
&\times \sum_{s=0}^n \left\{ \frac{(-n, -2\nu - n - 1)_s \Gamma(n + m + 2\nu - s - k + 3)}{\left(-\nu - n + i\frac{\beta}{2} \left(\frac{1}{\nu+n+1} + \frac{1}{\nu+m+1} \right) \right)_s s!} \right. \\
&\times \left. \frac{1}{\Gamma\left(\frac{n+m}{2} + \nu + 2 - s - i\frac{\beta}{2} \left(\frac{1}{\nu+n+1} + \frac{1}{\nu+m+1} \right) \right)} \right\}.
\end{aligned}$$

The proof is achieved by taking $n = m$.

Appendix B. Computation of $\phi_n^{(1,\nu,\beta)}$

From (3) we have

$$\begin{aligned} \frac{d}{dx}\phi_{n+1}^{(\nu,\beta)}(x) &= \frac{\pi K_{n+1}^{\nu,\beta}}{L} \left[\left((\nu + n + 2) \cos \frac{\pi x}{L} - \frac{\beta}{\nu + n + 2} \sin \frac{\pi x}{L} \right) \right. \\ &\times P_{n+1}^{(a_{n+1}, \bar{a}_{n+1})} \left(i \cot \frac{\pi x}{L} \right) + \frac{i(n + 2\nu + 2)}{2} \sin^{-1} \frac{\pi x}{L} \\ &\times P_n^{(a_{n+1}+1, \bar{a}_{n+1}+1)} \left(i \cot \frac{\pi x}{L} \right) \left. \right] \sin^{\nu+n+1} \frac{\pi x}{L} \exp \left(- \frac{\beta \pi x}{L(n + \nu + 2)} \right) \end{aligned} \quad (83)$$

and

$$\begin{aligned} W_{\nu,\beta}\phi_{n+1}^{(\nu,\beta)}(x) &= \frac{\hbar\pi K_{n+1}^{\nu,\beta}}{L} \left[\frac{\beta}{\nu + 1} \sin \frac{\pi x}{L} - (\nu + 1) \cos \frac{\pi x}{L} \right] \\ &\times \sin^{\nu+n+1} \frac{\pi x}{L} \exp \left[\frac{-\beta\pi L^{-1}x}{L(\nu + n + 2)} \right] P_{n+1}^{(a_{n+1}, \bar{a}_{n+1})} \left(i \cot \frac{\pi x}{L} \right). \end{aligned}$$

From the latter expression and (83), we have

$$\begin{aligned} A_{\nu,\beta}\phi_{n+1}^{(\nu,\beta)}(x) &= \frac{\pi\hbar K_{n+1}^{\nu,\beta}}{L} \left[(n + 1) \left[\cos \frac{\pi x}{L} + \frac{\beta \sin \frac{\pi x}{L}}{(\nu + 1)(\nu + n + 2)} \right] \right. \\ &\times P_{n+1}^{(a_{n+1}, \bar{a}_{n+1})} \left(i \cot \frac{\pi x}{L} \right) + \frac{i(n + 2\nu + 2)}{2} \sin^{-1} \frac{\pi x}{L} \\ &\times P_n^{(a_{n+1}+1, \bar{a}_{n+1}+1)} \left(i \cot \frac{\pi x}{L} \right) \left. \right] \sin^{\nu+n+1} \frac{\pi x}{L} \exp \left(- \frac{\beta \pi x}{L(n + \nu + 2)} \right). \end{aligned}$$

Let us determine $\cos \frac{\pi x}{L} + \frac{\beta}{(\nu+1)(\nu+n+2)} \sin \frac{\pi x}{L}$:

$$\begin{aligned} &\cos \frac{\pi x}{L} + \frac{\beta}{(\nu+1)(\nu+n+2)} \sin \frac{\pi x}{L} \\ &= \sqrt{1 + \frac{\beta^2}{[(\nu+1)(\nu+n+2)]^2}} \left[\frac{1}{\sqrt{1 + \frac{\beta^2}{[(\nu+1)(\nu+n+2)]^2}}} \cos \frac{\pi x}{L} \right. \\ &\quad \left. + \frac{\beta}{(\nu+1)(\nu+n+2)\sqrt{1 + \frac{\beta^2}{[(\nu+1)(\nu+n+2)]^2}}} \sin \frac{\pi x}{L} \right] \\ &= \sqrt{1 + \frac{\beta^2}{[(\nu+1)(\nu+n+2)]^2}} \left(\cos \frac{\pi x}{L} \cos \alpha_{\nu,\beta}(n) + \sin \frac{\pi x}{L} \sin \alpha_{\nu,\beta}(n) \right) \\ &= \sqrt{1 + \frac{\beta^2}{[(\nu+1)(\nu+n+2)]^2}} \cos \left(\frac{\pi x}{L} - \alpha_{\nu,\beta}(n) \right) \\ &= \sqrt{\frac{\Delta_{n+1}^0 E_0^{(\nu,\beta)}}{\varepsilon_0(2\nu + n + 3)}} \cos \left(\frac{\pi x}{L} - \alpha_{\nu,\beta}(n) \right). \end{aligned} \quad (84)$$

Finally,

$$\begin{aligned} A_{\nu,\beta}\phi_{n+1}^{(\nu,\beta)}(x) &= K_{n+1}^{\nu,\beta} \left[\sqrt{\frac{2M(n+1)^2 \Delta_{n+1}^0 E_0^{(\nu,\beta)}}{2\nu + n + 3}} \cos \left(\frac{\pi x}{L} - \alpha_{\nu,\beta}(n) \right) \right. \\ &\times P_{n+1}^{(a_{n+1}, \bar{a}_{n+1})} \left(i \cot \frac{\pi x}{L} \right) + \frac{i\pi\hbar(n + 2\nu + 2)}{2L \sin \frac{\pi x}{L}} P_n^{(a_{n+1}+1, \bar{a}_{n+1}+1)} \left(i \cot \frac{\pi x}{L} \right) \left. \right] \end{aligned}$$

$$\times \sin^{\nu+n+1} \frac{\pi x}{L} \exp\left(-\frac{\beta \pi x}{L(\nu+n+2)}\right),$$

with

$$\begin{aligned} \cos^2 \alpha_{\nu,\beta}(n) + \sin^2 \alpha_{\nu,\beta}(n) &= \frac{1}{1 + \frac{\beta^2}{[(\nu+1)(\nu+n+2)]^2}} + \frac{\beta^2}{\beta^2 + [(\nu+1)(\nu+n+2)]^2} \\ &= 1, \end{aligned}$$

and the constant $K_{n+1}^{\nu,\beta}$ defined in (5).

Appendix C. Computation of the normalization constant of CS

By using the definition

$$\begin{aligned} 1 &=: (|\eta_{q,p}^{[m,\nu,\beta]}\rangle, |\eta_{q,p}^{[m,\nu,\beta]}\rangle) \\ &= (\mathcal{R}_m^{(\nu,\beta)})^2 \int_0^L dx \exp\left(\frac{2W_{m,\nu,\beta}(q)x}{\hbar}\right) \overline{\phi_0^{(m,\nu,\beta)}(x)} \phi_0^{(m,\nu,\beta)}(x) \\ &= (\mathcal{R}_m^{(\nu,\beta)} K_m^{(\nu,\beta)})^2 \int_0^L dx \sin^{2\nu+2m+2} \frac{\pi x}{L} \exp\left(\frac{2W_{m,\nu,0}(q)x}{\hbar}\right) \\ &\quad \times \overline{P_m^{(a_m,\bar{a}_m)}\left(i \cot \frac{\pi x}{L}\right)} P_m^{(a_m,\bar{a}_m)}\left(i \cot \frac{\pi x}{L}\right) \\ &= (\mathcal{R}_m^{(\nu,\beta)} K_m^{(\nu,\beta)})^2 \frac{(\bar{a}_m+1)_m (a_m+1)_m}{(m!)^2} \sum_{k=0}^m \frac{(-m, a_m + \bar{a}_m + m + 1)_k}{(\bar{a}_m+1)_k k!} \\ &\quad \times \sum_{s=0}^m \frac{(-m, a_m + \bar{a}_m + m + 1)_s}{(a_m+1)_s s!} \times \mathcal{S}, \end{aligned} \tag{85}$$

with

$$\begin{aligned} \mathcal{S} &= 2^{-k-s} \int_0^L dx \left\{ \sin^{2\nu+2m+2} \frac{\pi x}{L} \exp\left(-\frac{2\pi(\nu+m+1)x}{L} \cot \frac{\pi q}{L}\right) \right. \\ &\quad \left. \left(1 + i \cot \frac{\pi x}{L}\right)^k \left(1 - i \cot \frac{\pi x}{L}\right)^s \right\} \\ &= 2^{-k-s} L e^{\frac{i(k-s)\pi}{2}} \int_0^1 dx \sin^{2\delta+2}(\pi x) e^{tx}, \end{aligned}$$

In [1] it is shown that

$$\int_0^1 dx \sin^{2\delta+2}(\pi x) e^{tx} = \frac{\Gamma(2\delta+3) e^{t/2}}{4^{\delta+1} \Gamma(\delta+2 + i\frac{t}{2\pi}) \Gamma(\delta+2 - i\frac{t}{2\pi})}, \quad \delta > -\frac{3}{2}.$$

Then the relation (85) becomes

$$\begin{aligned} \frac{1}{(\mathcal{R}_m^{(\nu,\beta)})^2} &= L (K_m^{(\nu,\beta)})^2 (2^{\nu+m+1} m!)^{-2} e^{-\pi(\nu+m+1) \cot \frac{\pi q}{L}} \left| \left(-\nu - m + i \frac{\beta}{\nu+m+1} \right)_m \right|^2 \\ &\quad \sum_{k=0}^m \frac{(-m, -2\nu - m - 1)_k}{(-m - \nu - \frac{i\beta}{\nu+m+1})_k \Gamma(m + \nu + 2 - k + i(\nu+m+1) \cot \frac{\pi q}{L}) k!} \\ &\quad \sum_{s=0}^m \frac{(-m, -2\nu - m - 1)_s \Gamma(2m + 2\nu - s - k + 3)}{(-m - \nu + \frac{i\beta}{\nu+m+1})_s \Gamma(m + \nu + 2 - s - i(\nu+m+1) \cot \frac{\pi q}{L}) s!}, \end{aligned}$$

where $K_m^{(\nu,\beta)}$ is given in (5).

Appendix D. Integral involved in the resolution of the identity

Here, in similar way as in [1], using the well-known Fourier transform ([7] p 520)

$$\forall k \in \mathbb{R}, \forall \nu > -1, \quad \int_{\mathbb{R}} \frac{e^{-itx}}{2\pi \cosh^{2\delta+2}(x)} dx = \frac{4^\delta \Gamma(\delta + 1 - i\frac{t}{2}) \Gamma(\delta + 1 + i\frac{t}{2})}{\pi \Gamma(2\delta + 2)}. \quad (86)$$

The inverse Fourier transform yields

$$\forall x \in \mathbb{R}, \forall \delta > -1, \quad \int_{\mathbb{R}} \frac{4^\delta \Gamma(\delta + 1 - i\frac{t}{2}) \Gamma(\delta + 1 + i\frac{t}{2})}{\pi \Gamma(2\delta + 2)} e^{ikx} dx = \frac{1}{\cosh^{2\delta+2}(x)}. \quad (87)$$

The analytical extension is unique; then, the above equality can be extended for $x \in \mathbb{C}$ with $\mathcal{I}m(x) \in] -\pi/2, \pi/2[$. By taking $u = ix, t \rightarrow \frac{t}{\pi}$ and $u = \pi x - \frac{\pi}{2}$, $\delta = m + \nu - \frac{k}{2} - \frac{s}{2}$, $t = -2\pi(\nu + m + 1) \cot \pi q - i\pi(k - s)$, we arrive at

$$\begin{aligned} & \int_{\mathbb{R}} \frac{\Gamma(m + \nu + 1 - k + i(m + \nu + 1) \cot \pi q) \Gamma(m + \nu + 1 - s - i(m + \nu + 1) \cot \pi q)}{\pi^2 \Gamma(2m + 2\nu - k - s + 2)} \\ & \quad \times \exp \left((\nu + m + 1) \cot \pi q (1 - 2x) \right) dq \\ & = \frac{4^{-\nu-m}}{\sin^{2\nu+2m+2}(\pi x) \left(\frac{1+i \cot(\pi x)}{2} \right)^k \left(\frac{1-i \cot(\pi x)}{2} \right)^s}, \quad m + n + \nu - \frac{s}{2} - \frac{k}{2} > -1. \end{aligned}$$

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